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Collisionless damping of plasma waves

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Abstract. Using the Weisskopf–Wigner theory of line broadening we derive an expression for the natural width of the Landau resonance $\omega - \boldsymbol{k} \cdot \boldsymbol{v} = 0$. The conditions for the validity of the conventional quantum perturbation theory of plasmons, and the collisionless damping due to nonresonant electrons are then examined.

1. Introduction

In the conventional classical plasma kinetic theory one generally derives the expression for the longitudinal dielectric coefficient of a plasma from the self-consistent Vlasov–Poisson equations. The collisionless damping of the plasma waves is proportional to the imaginary part of the dielectric coefficient appropriate to the retarded boundary conditions. From quantum theory Pines and Schrieffer (1962) have shown that the dielectric coefficient of a plasma can also be obtained from the Einstein A and B coefficients appropriate to the Cerenkov emission and absorption of plasmons. According to Pines and Schrieffer the collisionless damping of plasma waves is a consequence of a statistical balance between the induced emission and absorption of plasmons. In the golden rule approximation of the Einstein A and B coefficients one gets only a resonant interaction between the electrons and the plasma waves. Consequently, only the electrons whose velocities $\boldsymbol{v}$ satisfy the Landau resonance condition $\omega - \boldsymbol{k} \cdot \boldsymbol{v} = 0$ contribute to the collisionless damping of the plasma waves of frequency ω and wavevector $\boldsymbol{k}$. It is of course physically instructive to examine how the results (ie the expression for the collisionless damping) in the golden rule approximation get modified when one makes use of the Einstein A and B coefficients in the Weisskopf–Wigner approximation (ie when one makes use of the theory of line broadening to Cerenkov transitions in a plasma). We will show that, according to the Weisskopf–Wigner theory of line broadening, the resonance factor $\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v})$ of the results in the golden rule approximation gets replaced by the lorentzian factor $(\gamma_v/\pi)\{(\omega - \boldsymbol{k} \cdot \boldsymbol{v})^2 + \gamma_v^2\}^{-1}$, where γ_v is the reciprocal of the lifetime of the particle state $|\boldsymbol{v}\rangle$. We may point out that $\hbar\omega\gamma_v$ is the total rate of spontaneous emission of energy in the form of plasmons by an electron moving with a velocity $\boldsymbol{v}$. That is, $\hbar\omega\gamma_v = \partial(\frac{1}{2}\mu v^2)/\partial t = \mu \boldsymbol{v} \cdot (\partial \boldsymbol{v}/\partial t) \simeq \mu v^2/\tau_v$, where τ_v is the radiative slowing-down time of the particle of mass μ and velocity $\boldsymbol{v}$. Clearly $\gamma_v \ll \omega$ for the quantum perturbation theory of plasmons to be meaningful. This as we shall see later is consistent with the familiar notion that the appropriate fine structure constant or the electron–plasmon coupling constant $q^2/\hbar v_\phi \ll 1$ for the quantum perturbation theory of plasmons to be meaningful. Here q is the electronic charge and v_ϕ is the phase velocity of the plasmon. Furthermore, we will show that the collisionless damping due to nonresonant electrons is of the order of this coupling constant.

2. Derivation

We consider a gas of electrons of charge q, mass μ and number density N in a box of volume L^3. Let $|\boldsymbol{v}\rangle$ be the quantum state of one of the electrons and $|\boldsymbol{v}'\rangle$ be an energetically lower state. In the golden rule approximation, the transition probabilities for absorption j_A and emission j_E of a plasmon of momentum $\hbar\boldsymbol{k}$ and energy $\hbar\omega$ are given by (Pines and Schrieffer 1962, Tsytovich 1970, Walters and Harris 1968)

$$j_A(\boldsymbol{v};\boldsymbol{v}') = N_k M \delta_{\boldsymbol{v}',\boldsymbol{v}-\hbar\boldsymbol{k}/\mu}\delta\{\omega-\boldsymbol{k}\cdot(\boldsymbol{v}-\hbar\boldsymbol{k}/2\mu)\} \quad (1)$$

$$j_E(\boldsymbol{v}';\boldsymbol{v}) = (N_k+1) M \delta_{\boldsymbol{v}',\boldsymbol{v}-\hbar\boldsymbol{k}/\mu}\delta\{\omega-\boldsymbol{k}\cdot(\boldsymbol{v}-\hbar\boldsymbol{k}/2\mu)\}, \quad (2)$$

where $M = (4\pi^2 q^2\omega/L^3\hbar k^2)$, and N_k represents the number of plasmons. In the Weiss-kopf–Wigner approximation, one can easily show that the transition probabilities of (1) and (2) become (Heitler 1954)

$$j_A(\boldsymbol{v};\boldsymbol{v}') \simeq N_k M \delta_{\boldsymbol{v}',\boldsymbol{v}-\hbar\boldsymbol{k}/\mu}(\gamma_v/\pi)[\{\omega-\boldsymbol{k}\cdot(\boldsymbol{v}-\hbar\boldsymbol{k}/2\mu)\}^2+\gamma_v^2]^{-1} \quad (3)$$

and

$$j_E(\boldsymbol{v}';\boldsymbol{v}) \simeq (N_k+1) M \delta_{\boldsymbol{v}',\boldsymbol{v}-\hbar\boldsymbol{k}/\mu}(\gamma_v/\pi)[\{\omega-\boldsymbol{k}\cdot(\boldsymbol{v}-\hbar\boldsymbol{k}/2\mu)\}^2+\gamma_v^2]^{-1}, \quad (4)$$

respectively, where

$$\begin{aligned}\gamma_v &\simeq \sum_{\boldsymbol{k}} M\delta\{\omega-\boldsymbol{k}\cdot(\boldsymbol{v}-\hbar\boldsymbol{k}/2\mu)\} \\ &= (L/2\pi)^3\int_0^\infty \mathrm{d}k\, k^2 \int \mathrm{d}\theta\, 2\pi\sin\theta M\delta\{(\omega+\hbar k^2/2\mu)-kv\cos\theta\} \\ &= (q^2/\hbar v)\int_0^\infty \mathrm{d}k(\omega/k). \end{aligned} \quad (5)$$

Here θ is the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{v}$. For plasmons, ω is approximately the plasma frequency $\omega_0 = (4\pi N q^2/\mu)^{1/2}$ and hence γ_v of (5) is logarithmically divergent both at small and large k. However, since $\omega \simeq \omega_0 \simeq \boldsymbol{k}\cdot\boldsymbol{v} = kv\cos\theta$, $k \gtrsim (\omega_0/v)$, and since the plasmon wavelength must be larger than the plasma Debye length $k \lesssim (\omega_0/v_0)$, where $v_0 = (\kappa T/\mu)^{1/2}$. Here T is the temperature of the electron gas. Hence

$$\gamma_v \simeq \begin{cases} \omega_0(q^2/\hbar v)\ln(v/v_0) & \text{for } v \gtrsim v_0 \\ 0 & \text{for } v \lesssim v_0. \end{cases} \quad (6)$$

From (6), one can easily show that (γ_v/ω_0) is a maximum when $\ln(v/v_0) = 1$ (ie when $v \simeq 2{\cdot}7v_0$). Thus $(\gamma_v/\omega_0)_{\max} \simeq (\alpha c/2{\cdot}7v_0)$, where $\alpha = (q^2/\hbar c) \simeq \frac{1}{137}$. Hence, for plasmons the quantum perturbation theory is meaningful if and only if $(\gamma_v/\omega_0)_{\max} \ll 1$, that is, if and only if $v_0/c \gg \alpha/2{\cdot}7$. This means that any result of the quantum perturbation theory of the plasmons is meaningful if and only if the temperature of the electron gas is very much greater than 3·7 eV.

Let $NF(\boldsymbol{v})$ represent the number of electrons per unit volume which are in the quantum state $|\boldsymbol{v}\rangle$. By applying the principle of detailed balance for the transition probabilities per unit volume of emission and absorption, we get

$$\frac{\partial(N_k/L^3)}{\partial t} = \int \mathrm{d}\boldsymbol{v} N(F(\boldsymbol{v})j_E(\boldsymbol{v}';\boldsymbol{v}) - F(\boldsymbol{v}')j_A(\boldsymbol{v};\boldsymbol{v}')) \quad (7)$$

for a nondegenerate electron gas. If we assume $N_k = 0$ at $t = 0$, then the solution of (7) is of the form $N_k \propto \{1 - \exp(-2\gamma_k t)\}$, where

$$2\gamma_k = \int \mathrm{d}\boldsymbol{v}(L^3 NM)\frac{(\gamma_v/\pi)(F(\boldsymbol{v}-\hbar\boldsymbol{k}/\mu)-F(\boldsymbol{v}))}{\{\omega-\boldsymbol{k}\,.\,(\boldsymbol{v}-\hbar\boldsymbol{k}/2\mu)\}^2+\gamma_v^2} \tag{8}$$

in the Weisskopf–Wigner approximation. For $\hbar k \ll \mu v$, the approximate form of (8) may be written

$$\gamma_k \simeq -\frac{\omega\omega_0^2}{2k^2}\int \mathrm{d}\boldsymbol{v}\frac{\gamma_v \boldsymbol{k}\,.\,\nabla_v F(\boldsymbol{v})}{(\omega-\boldsymbol{k}\,.\,\boldsymbol{v})^2+\gamma_v^2}. \tag{9}$$

In the limit $\gamma_v \to 0$, γ_k of (9) reduces to the conventional Landau damping $\gamma_k^{\mathrm{L}} \sim -(\pi\omega\omega_0^2/2k^2)(\partial F/\partial v)_{v=\omega/k}$. This Landau damping γ_k^{L} of the plasmons is due to the resonant electrons which satisfy the condition $\omega - \boldsymbol{k}\,.\,\boldsymbol{v} = 0$. However, it is seen from (9) that the natural width of the Landau resonance is given by γ_v of (6).

3. Evaluation in the limiting cases

The γ_k of (9) is the collisionless damping of the plasma waves. In general the evaluation of the integral over $\boldsymbol{v}$ in (9) for γ_k is extremely difficult. Nevertheless, one can evaluate γ_k of (9) for the following two limiting cases. Case (i), when $(\gamma_v/\omega_0)_{\max} < (\gamma_k^{\mathrm{L}}/\omega_0)$, that is when the natural width of the Landau resonance is less than the conventional Landau damping. Case (ii), when $(\gamma_v/\omega_0)_{\max} > (\gamma_k^{\mathrm{L}}/\omega_0)$.

For $(\gamma_v/\omega_0)_{\max} < (\gamma_k^{\mathrm{L}}/\omega_0)$, as a first approximation we can assume that the dominant contribution to the integral over $\boldsymbol{v}$ for γ_k of (9) comes from the resonant electrons (ie from values of v in the neighborhood of the wave phase velocity ω/k). This approximation is good if $\partial F/\partial v$ does not change appreciably in the velocity interval $(\omega-\gamma_v)/k \lesssim v \lesssim (\omega+\gamma_v)/k$. Then γ_k of (9) is approximately equal to the conventional Landau damping γ_k^{L}.

For $(\gamma_v/\omega_0)_{\max} > (\gamma_k^{\mathrm{L}}/\omega_0)$, as a first approximation we can assume that the dominant contribution to the integral over $\boldsymbol{v}$ for γ_k of (9) comes from the nonresonant electrons whose velocities v lie in the range $v_0 \lesssim v \lesssim \omega/k$. Since $\omega \simeq \omega_0 > \boldsymbol{k}\,.\,\boldsymbol{v}$ and since $\omega_0 > \gamma_v$ we can write $\{(\omega-\boldsymbol{k}\,.\,\boldsymbol{v})^2+\gamma_v^2\}^{-1} \simeq (\omega-\boldsymbol{k}\,.\,\boldsymbol{v})^{-2} \simeq \omega^{-2}+\omega^{-3}(2\boldsymbol{k}\,.\,\boldsymbol{v})$. For a maxwellian distribution of velocities the approximate value of γ_k of (9) may be written

$$\gamma_k \simeq -\frac{2\omega_0^2}{\omega^2}\int_0^{\omega/k} \mathrm{d}v\gamma_v v\frac{\partial F}{\partial v} = (8\pi)^{-1/2}\left(\frac{\alpha\omega_0^3 c}{\omega^2 v_0}\right)g\left(\frac{\omega}{kv_0}\right) \tag{10}$$

where

$$g(x) = \int_1^{x^2} \mathrm{d}y(\ln y)\exp(-y/2). \tag{11}$$

In deriving the result of (10) from (9) we have made use of the fact that $\gamma_v \geqslant 0$ for all $\boldsymbol{v}$ regardless of whether $v \gtrless 0$, and that for a one-dimensional maxwellian distribution $\partial F/\partial v$ is an odd function of v. That is, the leading nonzero contribution to the nonresonant damping γ_k of (10) comes only from the term $\omega^{-3}(2\boldsymbol{k}\,.\,\boldsymbol{v})$ in the expansion $(\omega-\boldsymbol{k}\,.\,\boldsymbol{v})^{-2} \simeq \omega^{-2}+\omega^{-3}(2\boldsymbol{k}\,.\,\boldsymbol{v})$. We may point out that by a graphical solution of (11), it is relatively easy to show that $g(x)$ is of order unity for $x > 2$ and is less than unity for $x < 2$. Since for plasmons $\omega \simeq \omega_0$, it is readily seen from (10) that the collisionless damping due to nonresonant electrons (ie γ_k/ω_0 of (10)) is of the order of $(\gamma_v/\omega_0)_{\max}$. This result is hardly surprising since any line broadening mechanism (regardless of whether it is collisional or collisionless) should give rise to a damping which is of the order of the width of the lorentzian line. In our case the Weisskopf–Wigner broadening of the Cerenkov emission line is due to the damping force of the emitted radiation on the emitter itself. That is, this broadening is a consequence of the fact that the velocity of the emitting electron must decrease due to the emission process itself.

In (3) and (4) we have neglected a rather small term corresponding to the Lamb shift of the particle–wave resonance condition $\omega - \boldsymbol{k} \cdot \boldsymbol{v} \simeq 0$. This Lamb shift arises from the difference in the radiative self-energies of the quantum states $|\boldsymbol{v}\rangle$ and $|\boldsymbol{v}'\rangle$. However, one can show that the contribution to plasmon damping due to this Lamb shift is negligibly small if the number of electrons in a Debye sphere is very much greater than one.

We should perhaps point out that one can only use the concept of a transition probability per unit time for plasmons if and only if the time t of interest is such that $t \gg \omega_0^{-1}$, that is, if and only if $\gamma_k \ll \omega_0$ since for plasmons the time of interest $t \simeq \gamma_k^{-1}$. Furthermore, the results of the golden rule approximation (ie (1) and (2)) are valid if and only if this time $t \ll \gamma_v^{-1}$. However, the results of the Weisskopf–Wigner approximation (ie (3) and (4)) are valid for all $t \leqslant \gamma_v^{-1}$ (see Heitler 1954).

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